arXiv:hep-th/0402074v1 10 Feb 2004

ITEP-TH-08/04

Non RG logarithms via RG equations

Dmitry Malyshev[1]

February 2004

Abstract

We compute complete leading logarithms in Φ^4 theory with the help of Connes and Kreimer RG equations. These equations are defined in the Lie algebra dual to the Hopf algebra of graphs. The results are compared with calculations in parquet approximation. An interpretation of the new RG equations is discussed.

Keywords: Renormalization group, leading logarithms, Hopf algebra of graphs.

1 Introduction

One of the most subtle problems in QFT is the divergence of Feynmen integrals. This problem was solved by Bogolubov and Parasuk [1]. The process of consistent subtraction of divergences is called Bogolubov's R-operation. Recently Connes and Kreimer discovered that the R-operation has the structure of Hopf algebra of graphs [2, 3, 4].

The existence of the Hopf algebra has many interesting theoretical consequences such as connection to deformation quantization [5] and noncommutative geometry [6]. From the practical point of view the Hopf algebra sheds light on combinatorics of R-operation and simplifies the multiloop calculations [7, 8]. The Hopf algebra is also tightly connected to the renormalization group in QFT

[1]dmalyshe@princeton.edu

[4, 9]. In the present paper we utilize this connection for the calculation of the leading logarithms in Φ^4 theory. We will show that the Connes and Kreimer RG equations are more restrictive and enable one to find the leading logarithms in an arbitrary point (at least in some theories). The ordinary methods give the asymptotics only in symmetric points with respect to external momenta because in arbitrary points there are logarithms of ratios of external momenta invisible for RG equations.

The main object of consideration is the linear space of graphs $\mathcal{H}$. This space is infinite dimensional with basis labeled by graphs. Dual space $\mathcal{H}^*$ is the space of linear functions on $\mathcal{H}$. Any QFT may be interpreted as a vector $F \in \mathcal{H}^*$. If we know the couplings, then the function F maps the graphs to the corresponding Feynman integrals.

RG equations describe the change of couplings with the change of the scale. The set of couplings defines a vector $F \in \mathcal{H}^*$. The change of F is described by RG equations in the space $\mathcal{H}^*$. These equations have the form of linear differential equations. The transition from the ordinary RG equations to the RG equations in the space $\mathcal{H}^*$ seems to be trivial but it is a crucial step, because the last equations show up to be more informative. Indeed, a linear equation on a vector is a system of equations on the coordinates. In the case of graphs it means that there is an RG equation for any Feynman integral. What could be an interpretation of the new RG equations? The problem is that different Feynman diagrams with the same number of vertices have approximately the same magnitude and it seems that they are indistinguishable in experiments. But this is not the case when we have several particles with the same interaction but with different charges. In this case, Feynman integrals may give different contributions in different observables, and we can distinguish the integrals by comparing the observables. At the end we will consider an example of a theory with two charges.

The paper is organized as follows. In the next section we will state the generalized RG equations and discuss their properties. In section 3 we calculate the leading logarithms using the RG equations. In section 4 we compare the results with the calculations in parquet approximation. After that we study the RG equations in the model with two charges.

2 Generalized RG equations

In this section, we describe the RG equations in the linear space of graphs. The Hopf algebra of graphs appeared as a mathematical structure underlying the R-operation [3]. The Hopf algebra is dual to a Lie algebra of graphs. The RG equations are generated by some special element of the Lie algebra. This element is called the beta-function [4]. In the present paper we will focus on the application of RG equations to the case of leading logarithms in massless Φ^4 theory in $d = 4$.

Throughout the paper, we will consider only 1 particle irreducible diagrams with four external legs. Let γ_n be an n-loop diagram. We define $F(\gamma_n)$ to be the leading logarithm contribution to the diagram.

The ordinary RG equation (for the leading logarithms) is

$$\frac{\partial}{\partial \log \Lambda^2} F^{(n)} = \beta(g) \frac{\partial}{\partial g} F^{(n-1)}, \tag{1}$$

where $\beta(g)$ is the one-loop beta-function and $F^{(n)}$ is the n loop contribution to the four point function

$$F^{(n)} = \sum_{\gamma_n} \frac{g^{n+1}}{S_{\gamma_n}} F(\gamma_n), \tag{2}$$

where S_{γ_n} is the symmetry factor for the graph γ_n.

The RG equation in the space of linear functions of graphs is an equation on the vector $F \in \mathcal{H}^*$. The one-loop RG equation for the coordinates of this vector is

$$\frac{\partial}{\partial \log \Lambda^2} F(\gamma_n) = \sum_{\gamma_{n-1} = \gamma_n/\gamma_1} F(\gamma_{n-1}), \tag{3}$$

as before $F(\gamma_n)$ is the leading logarithm contribution for the graph γ_n. The symbol γ_n/γ_1 denotes the graph obtained by the contraction of subgraph $\gamma_1 \subset \gamma_n$ into a point. The sum is over $(n-1)$-loop graphs obtained by the contraction of one-loop subgraphs in γ_n. Note that in this equation we have no numerical factors, such as symmetry factors S_{γ_n}. For the explanation of 'miracle' cancellation of these factors see [10].

Equation (3) has two remarkable properties: it preserves the orientation of the graphs and it is linear in F.

Let us define the orientation in Φ^4 theory for the graphs contributing to the leading logarithms. All such graphs are 2 particle reducible [11], i.e. in any graph

there are 2 lines such that the cutting of these lines makes the graph disjoint. Let q denote the sum of the momenta flowing through the cut lines. Then q^2 is one of the three Mandelstam variables s, t or u. Consequently there are three possible orientations of the graphs which we will denote by the same letters s, t and u. The conservation of the orientation means that the graphs on both sides of equation (3) have the same orientation.

The linearity of equation (3) enables one to write it in the form of geodesic equation. We define the operator

$$\beta^{-1}\gamma_n = \sum_{\gamma_{n-1}=\gamma_n/\gamma_1} \gamma_{n-1}, \tag{4}$$

here γ_n, γ_{n-1} denote the basis vectors in $\mathcal{H}^*$ corresponding to graphs γ_n, γ_{n-1}. Recall that the operator β acts by insertion of one-loop subgraphs. The operator β^{-1} acts in the opposite way by contraction of one-loop subgraphs, though strictly speaking it is not the inverse to β. The non trivial components of β^{-1} are

$$(\beta^{-1})^{\gamma_{n-1}}_{\gamma_n} = \left[\begin{array}{ll} 1, & if \;\; \gamma_{n-1} = \gamma_n/\gamma_1; \\ 0, & if \;\; \gamma_{n-1} \neq \gamma_n/\gamma_1. \end{array} \right. \tag{5}$$

Equation (3) takes the form

$$\frac{\partial}{\partial \tau} F - F \circ \beta^{-1} = 0, \tag{6}$$

where $\tau = \log \Lambda^2$. In the components, this equation reads (we assume the summation over γ_{n-1})

$$\frac{\partial}{\partial \tau} F_{\gamma_n} - (\beta^{-1})^{\gamma_{n-1}}_{\gamma_n} F_{\gamma_{n-1}} = 0, \tag{7}$$

here $F_{\gamma_n} := F(\gamma_n)$. We see that in the space $\mathcal{H}^*$ the operator β^{-1} is a connection associated with the differentiation over $\tau = \log \Lambda^2$.

3 Leading logarithms

In this section, we find the leading logarithms in Φ^4 theory. The RG equations exist in any theory but the possibility to find the non RG logarithms is specific for the Φ^4 theory, because in the derivation we will use some specific properties of graphs. In this paper, we will not consider the Sudakov double logarithms and infrared divergences.

The derivation is divided into three steps. First, we consider the graphs which depend only on one external momentum, then the graphs depending on two momenta, and at the end we consider the general case.

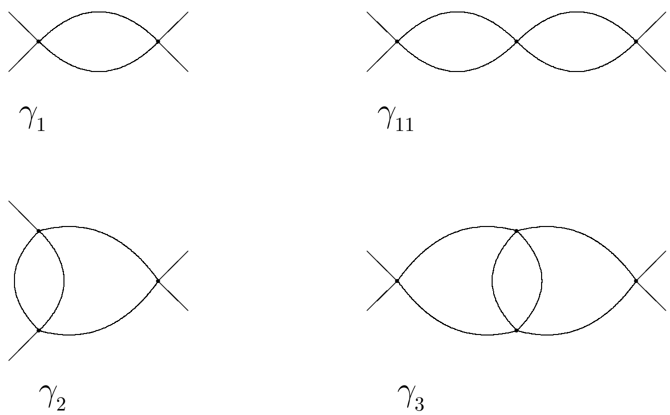

Figure 1: **Notations for graphs**

1. Let a graph γ_n depend only on one external momentum q. Then the only dimensionless combination is q^2/Λ^2, and the leading logarithm has the form

$$F(\gamma_n) = c(\gamma_n)\left(\log\frac{\Lambda^2}{q^2}\right)^n. \tag{8}$$

We find the leading logarithms for such graphs by induction. The result for the one-loop graph is

$$F(\gamma_1) = \log\frac{\Lambda^2}{q^2}. \tag{9}$$

Now we use equation (3) in order to express the coefficient $c(\gamma_n)$ in terms of the coefficients $c(\gamma_{n-1})$ for some $(n-1)$-loop graphs. The answer is as follows

$$c(\gamma_n) = \frac{1}{n}\sum_{\gamma_{n-1}=\gamma_n/\gamma_1} c(\gamma_{n-1}). \tag{10}$$

If we consider a symmetric point in external momenta, then this formula is valid for any graph γ_n [10]. For an arbitrary point, we also have to find the coefficients in front of the non RG logarithms. As an example of application of formula (10), we find the coefficients for the graphs γ_{11} and γ_3. In the graph γ_{11} we can contract

the left and the right one-loop subgraphs; each time the result of the subtraction is the graph γ_1

$$c(\gamma_{11}) = \frac{1}{2}(c(\gamma_1) + c(\gamma_1)) = 1. \tag{11}$$

In the graph γ_3 we can contract only the one-loop subgraph in the middle $\gamma_3/\gamma_1 = \gamma_{11}$

$$c(\gamma_3) = \frac{1}{3}c(\gamma_{11}) = \frac{1}{3}. \tag{12}$$

The leading logarithms are

$$F(\gamma_{11}) = \left(\log \frac{\Lambda^2}{q^2}\right)^2; \tag{13}$$

$$F(\gamma_3) = \frac{1}{3}\left(\log \frac{\Lambda^2}{q^2}\right)^3. \tag{14}$$

2. The next step is to consider the graphs that depend on two external momenta. We may assume that $p >> q$. (In the case $p << q$ we can put $p = 0$ and we get the previous case of a graph depending on one external momentum.)

In the case of two external momenta the leading logarithms have the form

$$F(\gamma_n) = \sum_{k=0}^{n} c_k(\gamma_n)\left(\log \frac{\Lambda^2}{q^2}\right)^{n-k}\left(\log \frac{p^2}{q^2}\right)^k. \tag{15}$$

If we know the leading logarithms for $(n-1)$-loop graphs, then we can use formula (3) and find the first n coefficients $c_0, \ldots, c_{n-1}$. The problem is with the last term

$$c_n(\gamma_n)\left(\log \frac{p^2}{q^2}\right)^n, \tag{16}$$

which vanishes as we differentiate with respect to $\log \Lambda^2$.

In order to find this last coefficient, we note that the integral over p is equal to some $(n+1)$-loop diagram which depends only on q

$$\int \frac{d^4p}{p^2(p+q)^2}F(\gamma_n; p, q) = F(\gamma_{n+1}; q). \tag{17}$$

In this equation, we know the right hand side, since it depends only on one external momentum.

The integral is effectively

$$\int \frac{d^4p}{p^2(p+q)^2} = \int_0^{\log \Lambda^2/q^2} d\log \frac{\Lambda^2}{p^2}. \tag{18}$$

After the integration we get the relation

$$\sum_{k=0}^{n} \frac{c_k(\gamma_n)}{k+1} = c(\gamma_{n+1}). \tag{19}$$

As an example, we find the leading logarithms for the graph γ_2.

$$F(\gamma_2) = c_0 \left(\log \frac{\Lambda^2}{q^2}\right)^2 + c_1 \left(\log \frac{\Lambda^2}{q^2}\right)\left(\log \frac{p^2}{q^2}\right) + c_2 \left(\log \frac{p^2}{q^2}\right)^2. \tag{20}$$

First we use equation (3) in order to find c_0 and c_1

$$\frac{\partial}{\partial \log \Lambda^2} F(\gamma_2) = F(\gamma_1) = \log \frac{\Lambda^2}{q^2}, \tag{21}$$

consequently

$$\begin{aligned} c_0 &= \frac{1}{2}; \\ c_1 &= 0. \end{aligned}$$

The integration over p gives the graph γ_3 with $c(\gamma_3) = 1/3$ (see equation (12)). Now we use (19) in order to find c_2

$$\frac{c_2}{3} + \frac{c_1}{2} + \frac{c_0}{1} = c(\gamma_3). \tag{22}$$

The result is

$$c_2 = -\frac{1}{2}. \tag{23}$$

The leading logarithms are

$$\begin{aligned} F(\gamma_2) &= \frac{1}{2}\left(\log \frac{\Lambda^2}{q^2}\right)^2 - \frac{1}{2}\left(\log \frac{p^2}{q^2}\right)^2 & (24) \\ &= \left(\log \frac{\Lambda^2}{q^2}\right)\left(\log \frac{\Lambda^2}{p^2}\right) - \frac{1}{2}\left(\log \frac{\Lambda^2}{p^2}\right)^2. & (25) \end{aligned}$$

3. The calculation in the general case is similar to the case of two external momenta. But now we have $(n+1)(n+2)/2$ coefficients for an n-loop diagram. Among these coefficients, $n+1$ correspond to terms without $\log \Lambda^2$ and are not fixed by RG equations. Integration over one of the external momenta gives an $n+1$-loop diagram with two external momenta. Since we already know the result for such diagrams, we can find the unknown $n+1$ coefficients. The only subtle

point here is to prove that the expressions in front of the coefficients do not vanish, i.e. that the analogue of equation (19) is not degenerate and we can find all $n+1$ coefficients.

We see that the leading logarithms in Φ^4 theory may be calculated with the help of Connes and Kreimer RG equations. The logic of this derivation may be reversed and we can calculate the leading logarithms using parquet approximation in order to check the new RG equations.

4 Parquet approximation

The computation of the leading logarithms in Φ^4 theory was performed by A.M.Polyakov in the appendix of the work [12]. The diagrams contributing to the leading logarithms have two-particle cross sections with three possible orientations [11]. Consequently, the vertex function has the following representation [12]

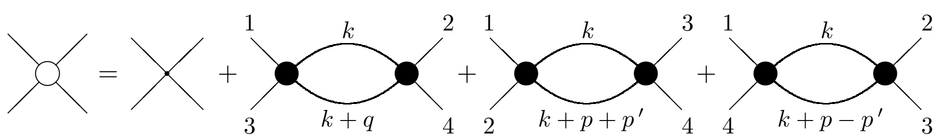

Figure 2: **Four-point vertex function**

The integration momentum k is chosen to be smaller than any other integration momentum in the shaded blocks. The external momenta are defined as follows

$$p_{1,3} = q/2 \pm p\ , \qquad p_{2,4} = -q/2 \pm p\,', \tag{26}$$

so that $s = (p+p\,')^2$, $t = q^2$ and $u = (p-p\,')^2$. We let $p\,' >> p >> q$ and denote

$$\xi = \log\frac{\Lambda^2}{p\,'^2}, \quad \zeta = \log\frac{\Lambda^2}{p^{\,2}}, \quad \eta = \log\frac{\Lambda^2}{q^2}. \tag{27}$$

This representation corresponds to the integral equation

$$F(\xi,\,\zeta,\,\eta) = -g + 3\int_0^{\xi} F^2(x,\,x,\,x)dx + \int_{\xi}^{\zeta} F(x,\,x,\,x)F(\xi,\,x,\,x)dx \\ + \int_{\zeta}^{\eta} F(\zeta,\,x,\,x)F(\xi,\,x,\,x)dx, \tag{28}$$

where the diagrams in s and u channels (the last two terms) correspond to

$$F_s(\xi) + F_u(\xi) = 2\int_0^{\xi} F^2(x,\, x,\, x)dx$$

and the diagram in t channel (the second term) gives

$$\begin{aligned} F_t(\xi,\, \zeta,\, \eta) = \int_0^{\xi} F^2(x,\, x,\, x)dx + \int_{\xi}^{\zeta} F(x,\, x,\, x)F(\xi,\, x,\, x)dx \\ + \int_{\zeta}^{\eta} F(\zeta,\, x,\, x)F(\xi,\, x,\, x)dx. \end{aligned} \tag{29}$$

The solution of equation (28) gives the summation of leading logarithms. We can use this equation in order to find the leading logarithms for individual Feynman integrals. We expand the vertex function in diagrams and substitute the expansion in the equation. Then we can find the leading logarithms recursively, since an n-loop diagram on the left hand side will be equal to some integrals of diagrams with no more than $(n-1)$-loops. For example, consider the diagram γ_2 in t channel that depends only on q and p, i.e. on η and ζ. Using equation (29), we find that this diagram equals

$$F_{\gamma_2}(\zeta,\, \eta) = \int_0^{\zeta} F_{\gamma_1}(x,\, x,\, x)F_{\gamma_0}dx + \int_{\zeta}^{\eta} F_{\gamma_1}(\zeta,\, x,\, x)F_{\gamma_0}dx, \tag{30}$$

where F_{γ_1} and F_{γ_0} correspond to one-loop and zero-loop diagrams, i.e. we insert the one-loop diagram in place of the left shaded block and a vertex instead of the right shaded block. The result is

$$F_{\gamma_2}(\zeta,\, \eta) = \int_0^{\zeta} x dx + \int_{\zeta}^{\eta} \zeta dx = \zeta\eta - \frac{1}{2}\zeta^2, \tag{31}$$

and we see that this result coincides with formula (25) obtained from the RG equations. Another example is the diagram γ_3 in t channel. This diagram has two possible decompositions: we can represent it as the diagram γ_2 in place of the left block and the vertex instead of the right block or we can put the vertex instead of the left block and γ_2 in place of the right block. The result is

$$\begin{aligned} F_{\gamma_3}(\eta) &= \int_0^{\eta} F_{\gamma_2}(x,\, x,\, x)F_{\gamma_0}dx + \int_0^{\eta} F_{\gamma_0}F_{\gamma_2}(x,\, x,\, x)dx \\ &= 2\int_0^{\eta} \frac{1}{2}x^2 dx = \frac{1}{3}\eta^3. \end{aligned}$$

This result should be compared with (12).

5 Two charge model

In this section, we make an attempt to find an interpretation of the new RG equation. Let us first revisit the logic of derivation of leading RG logarithms in QFT. During the subtraction of divergences, one has to introduce some scale necessary for the definition of the theory. Physical observables should not depend on this scale. This is achieved by the introduction of running couplings that depend on the scale in such a way that the observables are invariant. The possibility to do so imposes some restrictions on the theory. One of the consequences of these restrictions is that the leading logarithms are defined by the one-loop beta function. Thus the renormalizability of the theory fixes the form of the leading logarithms.

The question is whether the RG invariance of observables imposes some conditions on individual Feynman graphs. In a given process, only a sum of Feynman integrals is the observable amplitude. But in different processes, we may have different combinations of the integrals, and this may impose some constrains on the Feynman integrals. For example, consider the two-charge model

$$L(\varphi,\chi)=\frac{1}{2}(\partial\varphi)^2+\frac{1}{2}(\partial\chi)^2+\frac{g}{4!}\,\varphi^4+\frac{\tilde{g}}{4}\,\varphi^2\chi^2. \tag{32}$$

We will study two processes: $2\varphi\to2\varphi$ (two φ particles in two φ particles) and $2\chi\to2\varphi$ (two χ particles in two φ particles). Assume that $\tilde{g}<<g$. Then, in the second process, we neglect $\tilde{g}^2$. Additionally, we let the external momenta be almost the same, i.e. we will not study the non RG logarithms. In two-loop approximation we have

$$\begin{aligned}
F(2\varphi\to2\varphi) &= g-\frac{3}{2}g^2F(\gamma_1)+\frac{3}{4}g^3F(\gamma_{11})+\frac{6}{2}g^3F(\gamma_2);\\
F(2\chi\to2\varphi) &= \tilde{g}-\frac{1}{2}\tilde{g}gF(\gamma_1)+\frac{1}{4}\tilde{g}g^2F(\gamma_{11})+\frac{1}{2}\tilde{g}g^2F(\gamma_2).
\end{aligned}$$

In the second amplitude, we neglect the diagrams in t and u channels since they are proportional to $\tilde{g}^2$. The diagram γ_2 has only one orientation (out of 6 possible). The RG equations for the couplings are

$$\begin{aligned}
\frac{\partial}{\partial\log\Lambda^2}g &= \frac{3}{2}g^2;\\
\frac{\partial}{\partial\log\Lambda^2}\tilde{g} &= \frac{1}{2}g\tilde{g}.
\end{aligned} \tag{33}$$

In the leading logarithm approximation, we have

$$\begin{aligned} \frac{d}{d\log\Lambda^2} F(2\varphi \to 2\varphi) &= 0; \\ \frac{d}{d\log\Lambda^2} F(2\chi \to 2\varphi) &= 0. \end{aligned} \tag{34}$$

Taking into account equations (33), we rewrite the system (34) as

$$\begin{aligned} \partial_\tau F(\gamma_{11}) + 2\partial_\tau F(\gamma_2) &= 4F(\gamma_1); \\ \partial_\tau F(\gamma_{11}) + 4\partial_\tau F(\gamma_2) &= 6F(\gamma_1), \end{aligned}$$

where $\tau = \log\Lambda^2$. The solution of this system is

$$\begin{aligned} \partial_\tau F(\gamma_{11}) &= 2F(\gamma_1); \\ \partial_\tau F(\gamma_2) &= F(\gamma_1). \end{aligned}$$

We see that in the two charge model, the RG equations for the two-loop Feynman integrals take the form of Connes and Kreimer RG equations. This example suggests the following interpretation of RG equations (3). These RG equations ensure that, in any theory with several couplings and particles, the amplitudes will be RG invariant. Also it is plausible that the inverse statement is true, and there exists a theory with an infinite number of charges and particles such that the RG invariance of all the amplitudes in this theory is equivalent to RG equations (3).

Conclusion. In the paper, we have studied the RG equations for individual Feynman integrals. We argue that the RG invariance of any renormalizable theory should follow from equations of this type or, inversely, that the new RG equations follow from the RG invariance of all admissible theories. Using these equations, we calculate complete leading logarithms in Φ^4 theory and compare the result with parquet approximation.

The author is thankful to K.Selivanov and V.Belokurov for constant attention to the work, and to A.M.Polyakov for discussions and useful comments. The work is supported by RFBR grant 03-02-17373 and INTAS grant 00-334.

References

[1] N.N.Bogoliubov, D.V.Shirkov, Introduction to the Theory of Quantized Fields, 3rd ed., Wiley-Interscience, 1980.

[2] D. Kreimer, Adv.Theor.Math.Phys. 2 (1998) 303-334, q-alg/9707029

[3] A.Connes, D.Kreimer, Commun. Math. Phys. **210**, 249 (2000), hep-th/9912092.

[4] A.Connes, D.Kreimer, Commun. Math. Phys. **216**, 215 (2001), hep-th/0003188.

[5] L. M. Ionescu and M. Marsalli, hep-th/0307112.

[6] A. Connes and D. Kreimer, Commun. Math. Phys. **199**, 203 (1998), hep-th/9808042.

[7] D. Kreimer and R. Delbourgo, Phys. Rev. D **60**, 105025 (1999), hep-th/9903249.

[8] D. J. Broadhurst and D. Kreimer, Phys. Lett. B **475**, 63 (2000), hep-th/9912093.

[9] A.Gerasimov, A.Morozov, K.Selivanov, Int. J. Mod. Phys. A16: 1531 (2001), hep-th/0005053

[10] D. Malyshev, Phys. Lett. B **578**, 231 (2004), hep-th/0307301.

[11] I. T. Dyatlov, V. V. Sudakov, and K. A. Ter-Martirosyan, Sov. Phys. JETP **5**, 631 (1957), [Zh. Eksp. Teor. Fiz. **32**, 767 (1957)].

[12] A. M. Polyakov, Sov. Phys. JETP **30**, 151 (1970), [Zh. Eksp. Teor. Fiz. **57**, 271 (1969)].